Tide-induced groundwater head fluctuation in coastal multi-layered aquifer systems with a submarine outlet-capping

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Received 24 October 2006; received in revised form 16 January 2007; accepted 28 January 2007
Available online 3 February 2007

Abstract

This paper considered the tide-induced head fluctuations in two coastal multi-layered aquifer systems. Model I comprises two semi-permeable layers and a confined aquifer between them. Model II is a four-layered aquifer system including an unconfined aquifer, an upper semi-permeable layer, a confined aquifer and a lower semi-permeable layer. In each model, the submarine outlet of the confined aquifer is covered with a skin layer (“outlet-capping”). Analytical solutions of the two models are derived. In both models, leakages of the semi-permeable layers decrease the tidal head fluctuations. The outlet-capping reduces the aquifer’s head fluctuation by a constant factor and shifts the phase by a positive constant. The solution to Model II explains the inconsistency between the relatively small lag time and the strong amplitude damping effect of the tidal head fluctuations reported by Trefry and Johnston [Ground Water 1998;36:427–33] near the Port Adelaide River, Australia.

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Keywords: Tide; Coastal aquifer system; Elastic storage; Leakage; Outlet-capping; Analytical solution; Peaty layer; Aquifer parameter estimation; Least-squares method; Tidal head fluctuation

1. Introduction

The sea level variations have significant impacts on the groundwater flow in many coastal areas. Tidal fluctuation is one of the most important driving forces. The hydraulic head or water table in a coastal aquifer system usually fluctuates in response to tidal fluctuations. Many analytical researches have been conducted in this field since the middle period of the last century. For example, Jacob [2] provided the analytical solution of tide-induced groundwater level fluctuation in a single coastal confined aquifer; van der Kamp [14] considered a coastal aquifer which extends under

the sea infinitely; Li and Chen [5] investigated a coastal aquifer extending under the sea for a finite offshore length; Sun [11] developed an analytical solution in an estuary using a two-dimensional tidal loading boundary condition. The main limitation of these previous studies is that they only considered a single coastal aquifer configuration. In reality, most of the coastal aquifer systems are multi-layered, there is usually one or more confined aquifers and the semi-permeable layer(s) between them. Jiao and Tang [4] considered a coastal multi-layered aquifer system consisting of a confined aquifer, an unconfined aquifer, and a leaky layer between them. They ignored the leaky layer’s elastic storage. Li and Jiao [7] extended the result of Jiao and Tang [4] by taking the leaky layer’s elastic storage into account. Jeng et al. [3] presented a complete analytical solution to describe the tidal wave propagation and interference in the unconfined and confined aquifers separated by a thin, no-storativity leaky layer. Li and Jiao [8] considered a
coastal aquifer system including two aquifers separated by a semi-permeable layer, they improved previous work in that both the effects of the leaky layer’s elastic storage and the tidal wave interference between two aquifers were considered.

The semi-permeable layer’s elastic storage cannot be neglected when it is thick and has large specific storage, and may be even more important than that of the confined aquifer sometimes. According to various pumping test data available [7], if the semi-permeable layer is composed of thick, soft sedimentary materials, its elastic storage is much greater than that of the main aquifer. Most previous studies assumed that the coastal aquifer’s submarine outlet is directly connected to the seawater. The reality, however, is not always that case. Li and Chen [5,6] mentioned that the submarine outlet of a coastal aquifer may usually be covered by a thin silt-layer (outlet-capping). Li et al. [9] modeled this layer but their study only considered a single coastal aquifer.

This paper investigates the joint actions of the outlet-capping and the leakage and elastic storage of the semi-permeable layer in two commonplace coastal multi-layered aquifer systems. Model I consists of two semi-permeable layers and a confined aquifer between them. Model II is a four-layered aquifer system including an unconfined aquifer, an upper semi-permeable layer, a confined aquifer and a lower semi-permeable layer. In each model, the confined aquifer has a outlet-capping on the sea-land boundary at the coastline. Mathematical models are given for describing the two aquifer systems. Exact analytical solutions to the models are derived. The solutions are compared with existing solutions by previous researchers.

Trefry and Johnston [13] analyzed the pumping test data in a tidally influenced aquifer near the Port Adelaide River in Australia. Standard transient analyses based on the well-known Theis’s solution were first performed to the tidally corrected drawdown curves, yielding values of approximately 8–10 m/d for the hydraulic conductivity $K$ and 0.002 for the storativity $S$. Then the $S$-value and the Townley [12] solution were used to estimate the $K$-value based on the tidal effects. While the $K$-values (3–7 m/d) estimated from the tidal attenuation coefficients are roughly in line with the results of the pumping test analysis, those estimated from the tidal time lags (12–47 m/d) differ significantly from the pumping test analysis results, indicating a significant overestimation to the observed time-lags. Their numerical attempts using finite-element model to fit the tidal amplitudes and time lags in the observed drawdown curves by adjusting the storativity were also inconclusive. Based on these facts, they speculated that there may exist heterogeneities in the aquifer between the river bank and the test site.

In this paper, the new analytical solution is used to fit the observed tidal amplitudes and time lags in Trefry and Johnston [13]. The possibility of the existence of heterogeneities in the aquifer between the river bank and the test site is discussed, and the reason for the observed small time-lag is investigated.

2. Mathematical model and analytical solution

2.1. Mathematical model I

Model I is a coastal multi-layered aquifer system consisting of an upper and a lower semi-permeable layer and a confined aquifer between them (Fig. 1). The following assumptions will be used to the aquifer system:

1. The coastline is straight, all the layers are horizontal and extend landward infinitely, the bottom of the lower layer is impermeable.
2. Each of the three layers is homogeneous, isotropic and has constant thickness, the upper semi-permeable layer and the confined aquifer terminate at the coastline.
3. The flow is horizontal in the confined aquifer and vertical in the two semi-permeable layers [1,10].
4. The aquifer has a submarine outlet-capping, which may only cover the outlet of the middle confined aquifer as shown in Fig. 1, or also covers the submarine outcrops of the other two semi-permeable layers. Especially, the lower semi-permeable layer may also extend under the sea. These differences in the aquifer structure will not influence the mathematical model, as will be shown after the model equations below. The possibility of the existence of the submarine outlet-capping was discussed in Li et al. [9].
5. The density difference between the groundwater and the seawater can be neglected owing to its slight impact on groundwater level fluctuation [5].

Let the $x$-axis be positive landward and perpendicular to coastline, and the $z$-axis be vertical, positive upward (Fig. 1). The origin is located at the intersection of the coastline and the bottom of the lower semi-permeable layer. Using the assumptions above, the governing equations of the groundwater flow in the aquifer system can be written as

1. For the upper semi-permeable layer,

$$\frac{\partial h_1}{\partial t} = K_1 \frac{\partial^2 h_1}{\partial z^2}, \quad -\infty < t < \infty,$$

$$M + M_2 < z < M_1 + M + M_2, \quad (1)$$

$$\frac{\partial h_1}{\partial z} \bigg|_{z=M_1+M+M_2} = 0, \quad (2)$$

$$h_1(z, t; x) \bigg|_{z=M_1+M+M_2} = h(t, x), \quad (3)$$

where $h_1(z, t; x)$ denotes the hydraulic head [L] in the upper semi-permeable layer at the location $(x, z)$ and time $t$; $S_{s1}$ and $K_1$ represent the specific storativity [L$^{-1}$] and vertical hydraulic conductivity [LT$^{-1}$] of the upper semi-permeable layer, respectively; $M$, $M_1$ and $M_2$ are the thicknesses [L] of the confined aquifer, the upper and lower semi-permeable layers, respectively; $h(x, t)$ is the hydraulic head [L] of the confined aquifer at the instant $t$ and location $x$. 

2. For the lower semi-permeable layer,

$$S_{l2} \frac{\partial h_2}{\partial t} = K_2 \frac{\partial^2 h_2}{\partial z^2}, \quad -\infty < t < \infty, \quad 0 < z < M_2,$$

(4)

$$\frac{\partial h_2}{\partial z} \bigg|_{z=0} = 0,$$

(5)

$$h_2(z,t;x) \bigg|_{z=M_2} = h(t,x),$$

(6)

where $h_2(z,t;x)$ denotes the hydraulic head [L] in the lower semi-permeable layer at the location $(x,z)$ and time $t$. $S_{l2}$ and $K_2$ are the specific storativity [L$^{-1}$] and hydraulic conductivity [LT$^{-1}$] of the lower semi-permeable layer, respectively. One can see that Eqs. (1)-(6) remain the same if the semi-permeable layers are covered by other sediments at the coastline.

3. For the confined aquifer,

$$S \frac{\partial h}{\partial t} = T \frac{\partial^2 h}{\partial x^2} + K_1 \frac{\partial h}{\partial z} \bigg|_{z=M+M_2} - K_2 \frac{\partial h_2}{\partial z} \bigg|_{z=M_2},$$

$$-\infty < t < \infty, \quad x > 0,$$

(7)

$$\lim_{x \to \infty} \frac{\partial h}{\partial x} = 0,$$

(8)

where $S$ is the confined aquifer’s storativity (dimensionless) and $T$ the transmissivity [L$^2$T$^{-1}$]. Eq. (8) represents the no-flow boundary condition (the net recharge is zero far inland as $x$ approaches infinity).

Following Li et al. [9], the boundary condition of the outlet-capping can be expressed:

$$\frac{\partial h}{\partial x} \bigg|_{x=0} + K_m h(x,t) \bigg|_{x=0} = \frac{K_m}{mK} h,$$

(9)

where $K_m$ and $m$ are the hydraulic conductivity [LT$^{-1}$] and thickness of the outlet-capping [L], respectively; $K$ is the hydraulic conductivity [LT$^{-1}$] of the confined aquifer, $h_i = A \cos(\omega t)$ is the hydraulic head of the sea tide with $A$ being the tidal amplitude [L] and $\omega$ the tidal angular velocity [T$^{-1}$].

Due to space limitation, only the solution of the groundwater-head $h(x,t)$ in the confined aquifer will be given here. The derivation of $h(x,t)$ and the expressions of the heads in the upper and lower semi-permeable layers are detailed in Appendix A.

For the sake of convenience, six intermediate parameters are introduced. They are the confined aquifer’s tidal propagation parameter $a$ [L$^{-1}$], the dimensionless leakage of the outlet-capping $\sigma$, the semi-permeable layers’ dimensionless leakage $u_j$ and storativity ratio (dimensionless) $s_j$ defined as

$$a = \sqrt{\frac{\omega S}{2T}} = \sqrt{\frac{\pi S}{T_0}},$$

(10a)

$$\sigma = \frac{K_m}{amK},$$

(10b)

$$u_j = \frac{L_j}{S \omega M_j}, \quad j = 1, 2,$$

(10c)

$$s_j = \frac{S_j}{S} = \frac{M_{ij}}{S}, \quad j = 1, 2,$$

(10d)

where the subscripts $j = 1, 2$ represent the upper and lower semi-permeable layers, respectively.

Using these parameters above, the solution $h(x,t)$ can be written as

$$h(x,t) = AC e^{-\sigma t} \cos(\omega t - \frac{q_1 x}{p_1} - \frac{1}{\alpha^2} + \eta_1),$$

(11)

where $p_1$ and $q_1$ are two dimensionless constants defined as

$$p_1(s_1, u_1, s_2, u_2) = \int \left[ \zeta(s_1, u_1, s_2, u_2) + \eta_1(s_1, u_1, s_2, u_2) \right],$$

(12a)

$$q_1(s_1, u_1, s_2, u_2) = \frac{\zeta_1}{p_1},$$

(12b)

where the functions $\zeta_1$ and $\eta_1$ are given by

Fig. 1. Cross-section of a three-layered confined coastal aquifer system with a submarine outlet-capping (Model I).
with the intermediate parameters \( \theta_j \) defined as

\[
\theta_j = \sqrt{\frac{s_j}{2u_j}} = M_j \sqrt{\frac{\alpha S_{ij}}{2K_j}} \quad j = 1, 2,
\]

and the functions \( R_{th} \) and \( I_{th} \) defined as

\[
R_{th}(\theta) = \text{Re}\{\theta(1+i)\tanh[\theta(1+i)]\} = \theta \frac{1 - 2e^{-20 \sin 2\theta - e^{-40}}}{1 + 2e^{-20 \cos 2\theta + e^{-40}}}.
\]

\[
I_{th}(\theta) = \text{Im}\{\theta(1+i)\tanh[\theta(1+i)]\} = \theta \frac{1 + 2e^{-20 \sin 2\theta - e^{-40}}}{1 + 2e^{-20 \cos 2\theta + e^{-40}}}.
\]

The amplitude damping coefficient \( C_1 \) (dimensionless) and the constant phase shift \( \phi_l \) in (Radian) in Eq. (11), which are caused by the outlet-capping, are defined as

\[
C_1 = C_1(q_1, q_2, \sigma) = \frac{\sigma}{\sqrt{[q_1(s_1, u_1, s_2, u_2) + \sigma]^2 + q_2^2(s_1, u_1, s_2, u_2)}},
\]

\[
\phi_l = \phi_l(q_1, q_2, \sigma) = \arctan \left[ \frac{q_2(s_1, u_1, s_2, u_2)}{q_1(s_1, u_1, s_2, u_2) + \sigma} \right].
\]

### 2.2. Mathematical model II

Model II is a four-layered coastal aquifer system including an unconfined aquifer, an upper semi-permeable layer, a confined aquifer, and a lower semi-permeable layer. The aquifer has a submarine outlet-capping, as shown in Fig. 2. According to Jiao and Tang [4] and White and Roberts [15], assume that the shallow unconfined aquifer has a large specific yield which can damp effectively the tidal effect so that the tidal fluctuation in the unconfined aquifer is negligible compared to that in the confined aquifer. That is to say, the water table of the unconfined aquifer is a constant \( h_z \). The water table surface is chosen to be the datum of hydraulic-head so that \( h_z = 0 \). The other assumptions are the same as those of Model I.

Due to the existence of the unconfined aquifer, the upper boundary condition of hydraulic head \( h_1(z, t; x) \) in the upper semi-permeable layer is given by

\[
h_1(z, t; x) = M_{l1} + M_{l2} + M_s = h_s = 0.
\]

Namely, Eq. (2) in Model I is replaced by Eq. (17). All the other boundary conditions and governing equations of Model II are the same as those of Model I. The method of the derivation of the solution for Model II is similar to that for Model I so the details are omitted. The solution \( h(x, t) \) to the Model II reads

\[
h(x, t) = AC_1 e^{-\sigma x} \cos(ot - \alpha q_1 x - \phi_l),
\]

where \( p_{II} \) and \( q_{II} \) are defined as

\[
p_{II}(s_1, u_1, s_2, u_2) = \sqrt{\frac{\sigma^2(s_1, u_1, s_2, u_2) + \eta_{II}^2(s_1, u_1, s_2, u_2)}{\eta_{II}^2(s_1, u_1, s_2, u_2)}},
\]

\[
q_{II}(s_1, u_1, s_2, u_2) = \frac{\sigma^2(s_1, u_1, s_2, u_2) - \eta_{II}^2(s_1, u_1, s_2, u_2)}{\eta_{II}},
\]

where \( \xi_{II} \) and \( \eta_{II} \) are defined as

\[
\xi_{II}(s_1, u_1, s_2, u_2) = 1 + u_2 I_{th}(\theta_2) + u_1 I_{coh}(\theta_1),
\]

\[
\eta_{II}(s_1, u_1, s_2, u_2) = u_2 R_{th}(\theta_2) + u_1 R_{coh}(\theta_1).
\]
where

\[
R_{\text{coh}}(\theta) = \text{Re}\{\theta(1 + i) \coth[\theta(1 + i)]\} \\
= \theta 1 + 2e^{-2\theta} \sin \theta - e^{-\theta} \\
1 - 2e^{-2\theta} \cos \theta + e^{-\theta},
\]

\[
I_{\text{coh}}(\theta) = \text{Im}\{\theta(1 + i) \coth[\theta(1 + i)]\} \\
= \theta 1 - 2e^{-2\theta} \sin \theta - e^{-\theta} \\
1 - 2e^{-2\theta} \cos \theta + e^{-\theta}.
\]  

(21a)

(21b)

The amplitude damping coefficient (dimensionless) \(C_{uI}\) and the constant phase shift \(q_{uI}\) (in radian) caused by the outlet-capping are given by

\[
C_{uI} = C_{uI}(p_{uI}, q_{uI}, \sigma) = \frac{\sigma}{\sqrt[p_{uI}(s_1, u_1, s_2, u_2) + \sigma] + q_{uI}(s_1, u_1, s_2, u_2)},
\]

\[
q_{uI} = q_{uI}(p_{uI}, q_{uI}, \sigma) = \text{arctan} \left[\frac{q_{uI}(s_1, u_1, s_2, u_2)}{p_{uI}(s_1, u_1, s_2, u_2) + \sigma}\right].
\]  

(22)

(23)

3. Solution discussion and analysis

3.1. Comparisons with existing solutions

Li and Jiao [7] considered an aquifer system comprising a leaky confined aquifer, an unconfined aquifer and a semi-permeable layer between them. In their model, the confined aquifer is directly connected to the seawater at the coastline, i.e., there is no outlet-capping. It is obvious that this situation is a special case of Model II in this paper. In fact, if the lower semi-permeable layer in Model II is replaced by an impermeable layer \(K_2 = 0\) or equivalently \(u_2 = 0\), and let the thickness of the outlet-capping equal zero \(m \to 0\) or equivalently \(\lim_{m \to 0} \sigma = \infty\), then the solution (18) can be simplified into the Li and Jiao [7] solution. Here we only point out this fact and the details for the solution simplification will not be shown due to space limitation. Because the solutions by Jiao and Tang [4] and Jacob [2] are special cases of Li and Jiao [7], they are also special cases of the new solution (18).

3.2. Effects of the outlet-capping

**Model I**: Solution (11), (15) and (16) demonstrate that the existence of the submarine outlet-capping reduces the head fluctuation amplitude by a constant factor \(C_{1} (0 < C_{1} \leq 1)\) and leads to a constant phase shift \(q_{1} (0 \leq q_{1} \leq \pi/4)\), which corresponds to a time-lag within 3 h for the diurnal sea tide and 1.5 h for semi-diurnal tide. Here for estimating the maximum of \(q_{1}\) the inequality \(q_{1} \leq p_{1}\) is used, which is obvious from their definition Eqs. (12a) and (12b). When the outlet-capping does not exist (i.e., its thickness \(m \to 0\) or equivalently \(\lim_{m \to 0} \sigma = \infty\)), the factor \(C_{1} = 1\) and the phase shift \(q_{1} = 0\). Eq. (15) indicates that the factor \(C_{1}\) is an increasing function of the leakage \(\sigma\), and Eq. (16) indicates that the phase shift \(q_{1}\) is a decreasing function of \(\sigma\).

**Model II**: The same analysis as above can be made using the solution (18) of Model II, and the definition Eqs. (22) and (23) of \(C_{uI}\) and \(q_{uI}\), and the similar conclusions can be drawn. The outlet-capping reduces the head fluctuation amplitude by a constant factor \(C_{uI} (0 < C_{uI} \leq 1)\) and leads to a constant phase shift \(q_{uI} (0 \leq q_{uI} \leq \pi/4)\). For aquifer systems without the outlet-capping, the factor \(C_{uI} = 1\) and the phase shift \(q_{uI} = 0\). The factor \(C_{uI}\) increases with \(\sigma\), and the phase shift \(q_{uI}\) is a decreasing function of \(\sigma\).

3.3. Effects of the leakage and elastic storage of the semi-permeable layers

In the following discussion, the parameter value ranges \(0 \leq s_1, s_2 \leq 100, 0 \leq u_1, u_2 \leq 40\) will be used. These ranges are based on the discussion in Li and Jiao [7].

**Model I**: Solution (11), and the parameters defined by Eqs. (12)–(16) demonstrate that the head fluctuation amplitude relative to the tidal amplitude \(A\) is given by the term \(C_{I}(p_{I}, q_{I}, \sigma) e^{-q_{I}x}\). The greater the parameters \(p_{I}\) and \(q_{I}\), the smaller both the quantities \(C_{I}\) and \(e^{-q_{I}x}\). Therefore, it is important to know how \(p_{I}\) and \(q_{I}\) depend on the primary model parameters \(s_1, u_1, s_2, u_2\). Because \(p_{I}\) and \(q_{I}\) have the following symmetric properties with respect to the two pairs of parameters \((s_1, u_1)\) and \((s_2, u_2)\) (see Eqs. (12)–(14):

\[
p_{I}(s_1, u_1, s_2, u_2) = p_{I}(s_2, u_2, s_1, u_1),
\]

\[
q_{I}(s_1, u_1, s_2, u_2) = q_{I}(s_2, u_2, s_1, u_1),
\]  

(24)

it is enough to only analyze how \(p_{I}\) and \(q_{I}\) depend on the second pair of parameters \(s_2, u_2\) (lower semi-permeable layer) while the first pair of parameters are fixed. Fig. 3 shows how \(p_{I}\) changes with \(\lg u_2\) for different values of \(s_2\) when \(s_1\) and \(u_1\) vanish. One can see that \(p_{I}\) is always equal to or greater than 1 and increases with the storativity ratio \(s_2/p_{1}\) also increases with \(u_2\) when \(u_2\) is small. For large values of \(u_2\) and \(s_2\), \(p_{I}\) is significantly greater than 1, although it may not keep increasing with \(u_2\), indicating a significant damping effect on the tidal head fluctuation by reducing

![Fig. 3. Changes of the parameter \(p_{I}\) with \(\lg u_2\) for different values of \(s_2\) when \(s_1\) and \(u_1\) vanish.](image-url)
both $C_1$ and $e^{-\alpha_0 s_1}$. Fig. 4 shows how $q_1$ changes with $\lg u_2$ for different values of $s_2$ when $s_1$ and $u_1$ vanish. One can see that $q_1$ is always equal to or greater than 1 and increases with the leakance $u_2$. $q_1$ also increases with the storativity ratio $s_2$ for small values of $u_2$. For large values of $u_2$ and $s_2$, $q_1$ is significantly greater than 1, indicating a significant damping effect on the tidal head fluctuation by reducing the coefficient $C_1$. For large values of $u_2$, the curves in Fig. 4 cross each other. This implies that $q_1$ is not monotonic with respect to $s_2$. For very small values of $s_2 (s_2 \leq 0.1)$, both $p_1$ and $q_1$ are very close to 1. In fact, we have

$$\lim_{s_1, s_2 \rightarrow 0} p_1 = \lim_{s_1, s_2 \rightarrow 0} q_1 = 1. \quad (25)$$

Physically, because the lower semi-permeable layer has an impermeable bottom, the leakage flow is controlled by its elastic storage. When $s_2$ is small, no significant leakage flow will happen, no matter how great the leakance $u_2$ is.

In short, the leakage and elastic storage of the semi-permeable layer will decrease the tidal head fluctuation by means of reducing both $C_1$ and $e^{-\alpha_0 s_1}$. The existence of the outlet-capping reinforces significantly the damping effect of the semi-permeable layers on the tidal head fluctuations by reducing the amplitude coefficient $C_1$.

From solution (11), the phase shift of the tidal head fluctuation at an inland location $x$ is given by $aq_1 x + \phi_1$, which corresponds to a time lag $(aq_1 x + \phi_1)/\omega$. A similar analysis based on Figs. 3 and 4 concludes that the effective leakage and elastic storage of the semi-permeable layer will lead to values of $p_1$ and $q_1$ significantly greater than 1, which in turn results in considerable phase shift (or time lag). The existence of the outlet-capping increases the phase shift by a location-independent constant $\phi_1$.

**Model II: Solution (18)**, and the parameters defined by Eqs. (19)–(23) demonstrate that the head fluctuation amplitude relative to the tidal amplitude $A$ is given by the term $C_{II}(p_{II}, q_{II}, \sigma)e^{-\alpha_0 s_1}$. The greater the parameters $p_{II}$ and $q_{II}$, the smaller both the quantities $C_{II}$ and $e^{-\alpha_0 s_1}$. Therefore, it is important to know how $p_{II}$ and $q_{II}$ depend on the primary model parameters $s_1$, $u_1$, $s_2$ and $u_2$. The dependence of $p_{II}$ and $q_{II}$ on the second pair of parameters $(s_2, u_2)$ is the same as that of $p_1$ and $q_1$ as is demonstrated in Figs. 3 and 4. Therefore, it is enough to only analyze how $p_{II}$ and $q_{II}$ change with the first pair of parameters $s_1$, $u_1$ (upper semi-permeable layer) while the second pair of parameters $(s_2, u_2)$ are fixed. The case when $u_2$ and $s_2$ vanish was considered by Li and Jiao [7] (Figs. 2 and 3 in their paper, where the parameters $p$, $q$, $u$ and $s$ are $p_{II}$, $q_{II}$, $u_1$ and $s_1$ here, respectively, with the $u$-abscissa instead of $\lg u$). Here the dependence of $p_{II}$ and $q_{II}$ on $s_1$ and $u_1$ will be discussed with $s_2 = 10.0$ and $u_2 = 0.1$.

Fig. 5 shows how $p_{II}$ changes with $\lg u_1$ for different values of $s_1$ when $s_2 = 10.0$ and $u_2 = 0.1$. One can see that $p_{II}$ is always greater than 1 and increases with $u_1$ and $s_1$. For large values of $u_1$ and $s_1$, $p_{II}$ is significantly greater than 1, indicating a significant damping effect on the tidal head fluctuation by reducing both $C_{II}$ and $e^{-\alpha_0 s_1}$.

Fig. 6 shows how $q_{II}$ changes with $\lg u_1$ for different values of $s_1$ when $s_2 = 10.0$ and $u_2 = 0.1$. One can see that $q_{II}$ increases with the storativity ratio $s_1$ for any values of $u_1$. Although $q_{II}$ may not be a monotone function of $u_1$, it is nonnegative for any values of $u_1$ and $s_1$, indicating a damping effect on the tidal head fluctuation by reducing the coefficient $C_{II}$ (see Eq. (22)).

In short, the leakage and elastic storage of the upper semi-permeable layer in Model II will decrease the tidal head fluctuation by reducing both $C_{II}$ and $e^{-\alpha_0 s_1}$. The existence of the outlet-capping reinforces significantly the damping effect of the upper semi-permeable layer on the tidal head fluctuation by reducing the amplitude coefficient $C_{II}$.

From solution (18), the phase shift of the tidal head fluctuation at an inland location $x$ is given by $aq_{II} x + \phi_{II}$, which corresponds to a time lag $(aq_{II} x + \phi_{II})/\omega$. For aquifers with small $s_1$-values and large $u_1$-values, $q_{II}$ is significantly less than 1 (Fig. 6), while $p_{II}$ is significantly greater than 1 (Fig. 5). The small value of $q_{II}$ and large value of

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**Fig. 4.** Changes of the parameter $q_1$ with $\lg u_2$ for different values of $s_2$ when $s_1$ and $u_1$ vanish.

**Fig. 5.** Changes of the parameter $p_{II}$ with $\lg u_1$ for different values of $s_1$ when $s_2 = 10$ and $u_2 = 0.1$.**
the western bank of the Port Adelaide River (see Table 1). The wells had 50 mm diameters and were screened to 3.5 m below the ambient water table. Water level time series measured by piezometers in the monitoring wells exhibited regular semi-diurnal fluctuation, with amplitudes of the order of several centimeters. Tidal fluctuation amplitude in the river is about 1 m with a dominant semi-diurnal component of period 0.6 days. Trefry and Johnston [13] used the least-squares techniques to determine the best-fit attenuation coefficients (\(z\)) and lag times (\(\tau\)) for the tidal fluctuations in each monitoring well (see Table 1). Then the least-squares attenuations and phases were used to correct measured pumping test drawdowns for the standard Theis type curves analyses. Their analyses yielded values of approximately 7.8–10.3 m/d for \(K\) (hydraulic conductivity) and 0.0021–0.0026 for \(S\) (storativity) (Table 1). Because they obtained the storativity \(S\) with an aquifer thickness of 7.2 m, the specific storage of the aquifer is \(S_s = S/7.2 \text{ m}^{-1}\). The estimated low storativity value is significantly less than the specific yield of the aquifer, indicating a semi-confined condition. It is obvious that this semi-confined condition is not caused by the upper clay layer because the bottom of upper clay layer (0.7 m AHD) is higher than the ambient water table (approximately 0.45 m AHD) during the pumping tests. Therefore, the peaty layers seem to play an important role in forming the confinement condition in the aquifer. Due to this reason, it is reasonable to regard the peaty layer immediately above the tips of the monitoring wells (about \(-3.05 \text{ m AHD}\)) as a semi-permeable layer, the fine sand above and below which forms an unconfined and confined aquifer, respectively. The clay layer below the fine sand is regarded as an impermeable bottom (see Fig. 7). Consequently, the aquifer system is conceptualized into a three-layered aquifer system consisting of an unconfined aquifer, a confined aquifer, and a semi-permeable layer.

4. Application

4.1. Hydrogeological background and conceptual model

Trefry and Johnston [13] analyzed the pumping test data in a tidally influenced aquifer near the Port Adelaide River in Australia. The study area is situated in a shallow surficial aquifer on the western bank of the Port Adelaide River, approximately 10 km upstream from the river mouth.

The ground surface in the vicinity of the pumping test bores is located at 2.44 m AHD (Australia Height Datum), with the corresponding annual mean water table position at 0.30 m AHD. The formations of aquifer vary in the sequence from top to bottom. The top layer is composed of loose fill. Next in the sequence is a 0.3 m thick clay layer with its bottom at the elevation of 0.7 m AHD. The lower layer is the bulk aquifer with peaty layers (several centimeters thick) in the vertical. The aquifer is underlain by a clay layer at \(-6.8 \text{ m AHD}\), this provides a uniform impermeable bottom to the aquifer system throughout the study area (see Fig. 7). All the geologic information here is taken from Trefry and Johnston [13].

The test data were collected from the seven monitoring wells (BC5, BC6, BC11-15), which are 120–200 m far from the outlet-capping increases the phase shift by a location-independent constant \(q_{\text{lim}}\).

When the upper confining layer becomes impermeable, i.e., \(u_1 \rightarrow 0 (|\log u_1| \ll 0)\), \(p_{\text{III}}\) and \(q_{\text{III}}\) tends to constants greater than 1 for any \(s_1\). In fact, we have

\[
\lim_{u_1 \to 0} p_{\text{III}}(|\log u_1| = 0.1) = p_{\text{III}}(0, 0, 10, 0.1) \approx 1.598, \tag{26}
\]

\[
\lim_{u_1 \to 0} q_{\text{III}}(|\log u_1| = 0.1) = q_{\text{III}}(0, 0, 0, 0) \approx 1.068. \tag{27}
\]

In this case the tidal head fluctuation in the aquifer is influenced only by the lower semi-permeable layer and the outlet-capping as discussed above for Model I.

Fig. 6. Changes of the parameter \(q_{\text{III}}\) with \(\log u_1\) for different values of \(s_1\) when \(s_2 = 10\) and \(u_3 = 0.1\).
test site, which leads to the inconsistency between the estimations based on the attenuation coefficient and the lag time. In order to verify this speculation, we assume that the outflow of confined aquifer at the river bank boundary is covered by a skin-layer (e.g. silt-layer) with permeability different from that of the aquifer at the test site. The aquifer system at the test site can then be described by Model II and solution (18) can be used to predict the tide-induced head fluctuation at tips of the monitoring wells. The impermeable lower clay layer ($w_2 = 0$) implies that the solution is independent of the parameters $s_2$ and $u_2$ (see Eqs. (19) and (20)).

4.2. Aquifer parameter estimation

Using Eq. (10a) and $S = S/7.2$ m$^{-1}$, one has

$$\alpha = \sqrt{\pi S/(Th_0)} = \sqrt{\pi S/(Kt_0)} = \sqrt{\pi S/(7.2Kt_0)}$$

(28)

Based on Eq. (28), the data of $t_0 = 0.6$ days and the $K$ and $S$ values given by Trefry and Johnston [13] (see Table 1) were used to estimate the tidal propagation parameter values at the four wells BC12-15, which are listed in the column beginning with $a_0$ in Table 1. For the other three wells (BC5, BC6 and BC11) the value of $a_0$ is obtained using the averages of the $K$- and $S$-values at the four wells BC12-15.

The amplitude attenuation coefficient $\alpha$ and lag time $\tau$ predicted by Model II (solution (18)) at each monitoring well is

$$\alpha_{II}(x_w; a, s_1, u_1, \sigma) = C_{II}(p_{II}, q_{II}, \sigma) \exp\left[-a p_{II} x_w\right]$$

$$\tau_{II}(x_w, a, s_1, u_1, \sigma) = \frac{a q_{II} x_w + \phi_{II}(p_{II}, q_{II}, \sigma)}{\omega}$$

(29a)\hspace{1cm} (29b)

with $x_w$ being the distance from the well “w” to the river bank. To estimate the four unknown parameters $a$, $s_1$, $u_1$ and $\sigma$, the following least-squares objective function

$$L_{SR}(a, s_1, u_1, \sigma) = (a - a_0)^2 + \left[\tau_w - \tau_{II}(x_w; a, s_1, u_1, \sigma)\right]^2$$

$$+ \left[\tau_s - \tau_{II}(a, s_1, u_1, \sigma)\right]^2$$

$$(w = BC5, BC6, BC11 - 15)$$

(30)
was minimized with respect to $a$, $s_1$, $u_1$ and $\sigma$ for each of the seven wells. Here $z_w$ is the attenuation coefficient and $\tau_w$ the lag time at the well "w". The penalty term $(a - a_0)^2$ is used to incorporate a priori knowledge about the tidal propagation parameter obtained from the existing data of Trefry and Johnston [13] in a flexible way. A Fortran code was developed to solve numerically the least-squares problem (30) using quasi-Newton iteration method. Because the peaty layer is very thin (less than 0.1 m), its storativity is much less than that of the aquifer, which will lead to a small storativity ratio $s_1$. Due to this, small initial values of $s_1$ between 0.001 and 0.2 were used when minimizing the objective function (30). The minimizing results are less sensitive to the change of $s_1$ in the sense that the final $s_1$-value remains approximately the same as the initial value and that the variations of the other estimated parameters such as $a$, $u_1$ and $\sigma$ are slight when $s_1$ changes from 0.001 to 0.2 (see Table 1). The initial value of $a$ is set to be $a_0$. Different initial values of $u_1$ and $\sigma$ were used and the finally converged values are listed in Table 1. The finally converged values of $a$ may be different but all of them are greater than 1000.0, which always leads to $C_{11} = 1.00$ and $\varphi_{11} = 0.00$ and indicates that the speculated skin layer at the river bank does not exist, i.e., the confined aquifer is homogeneous in horizontal direction. For each well, the fitted attenuation coefficient and lag time by minimizing (30) are always the same as their respective values determined by Trefry and Johnston [13]. The estimated tidal propagation parameter $a$ by minimizing (30) is slightly different from its initial estimation $a_0$ which is based on the Theis analysis results by Trefry and Johnston [13]. The estimated dimensionless leakages of the peaty layer at the seven wells range from 0.56 to 1.62 with an average of 1.10. The minimums of the objective function (30) range from 3.67$^{-7}$ to 1.99$^{-5}$. All the details are given in Table 1. These results show that our analytical solution fits the observed tidal attenuation coefficients and the lag times at the seven wells satisfactorily, the estimated aquifer parameter values are also within their reasonable value ranges.

5. Conclusions

This paper considered the tide-induced head fluctuations in two coastal multi-layered aquifer systems. Model I comprises two semi-permeable layers and a confined aquifer between them. Model II is a four-layered aquifer system including an unconfined aquifer, an upper semi-permeable layer, a confined aquifer and a lower semi-permeable layer. The confined aquifer in each model has a submarine outlet-capping. Exact analytical solutions of the two models are derived. In both models, leakages of the semi-permeable layers decrease the tidal head fluctuations in the aquifers. For the case that the upper semi-permeable layer is covered by an unconfined aquifer and has a small elastic storage, large leakage through the semi-permeable layer may lead to short lag time. For all the other cases, large leakage results in long lag time. The outlet-capping reinforces the damping effect of the semi-permeable layers on the head fluctuation in the aquifer by a constant reduction factor and shifts the phase by a positive constant. The solution to Model II is used to explain the inconsistency between the relatively small lag time and the strong amplitude damping effect reported by Trefry and Johnston [13] in seven monitoring wells screened in a coastal confined aquifer near the Port Adelaide River, Australia. The result demonstrated that our theoretical model incorporating the leakage effect of the peaty layer fits satisfactorily the observed attenuation coefficients and lag times.

Acknowledgements

This research is supported by the National Natural Science Foundation of China (No. 40672167) and Academic Exploration and Innovation Foundation for Graduate Students in China University of Geosciences in the years of 2005–2006.

Appendix A. Derivation of the solution

Suppose

$$h = A \text{Re}[X(x) \exp(i \omega t)],$$  \hspace{1cm} (A.1)

$$h_1 = A \text{Re}[Z_1(z)X(x) \exp(i \omega t)],$$  \hspace{1cm} (A.2)

$$h_2 = A \text{Re}[Z_2(z)X(x) \exp(i \omega t)],$$  \hspace{1cm} (A.3)

where $X(x)$, $Z_1(z)$, $Z_2(z)$ are complex functions, $\text{Re}$ denotes the real part of the followed complex expression, $i = \sqrt{-1}$. Substituting (A.2) back into Eqs. (1)–(3), and then extending the three resultant real equations into complex ones with respect to the unknown function $Z_1(z)$, yield

$$Z_1'(z) - i \alpha S_{11} Z_1(z) = 0,$$  \hspace{1cm} (A.4)

$$Z_1'(M_1 + M + M_2) = 0,$$  \hspace{1cm} (A.5)

$$Z_1(M + M_2) = 1.$$  \hspace{1cm} (A.6)

The solution of (A.4)–(A.6) is

$$Z_1(z) = \frac{\exp \left[\frac{\alpha}{M_1}(1+i)(2M_1 + 2M + 2M_2 - z)\right] + \exp \left[\frac{\alpha}{M_1}(1+i)z\right]}{\exp \left[\frac{\alpha}{M_1}(1+i)(2M_1 + M + M_2)\right] + \exp \left[\frac{\alpha}{M_1}(1+i)(M + M_2)\right]},$$  \hspace{1cm} (A.7)

Using (A.7), one can obtain

$$Z_1'(M + M_2) = - \frac{1}{M_1}(1+i)\theta_1 \tanh[(1+i)\theta_1]$$

$$= - \frac{1}{M_1}(1+i)\theta_1 \frac{\sinh[(1+i)\theta_1]}{\cosh[(1+i)\theta_1]}$$

$$= \frac{1}{M_1}(R_{\theta_1}(\theta_1) + i I_{\theta_1}(\theta_1)),$$  \hspace{1cm} (A.8)

where $\theta_1$ is given by (13c), and the functions of $R_{\theta_1}(\theta)$ and $I_{\theta_1}(\theta)$ by Eqs. (14a) and (14b), respectively.
Substituting Eq. (A.3) back into Eqs. (4)–(6), and then extending the three resultant real equations into complex ones with respect to the unknown function \( Z_2(z) \), yield
\[
Z_2''(z) - \frac{i\alpha S_2}{K_2} Z_2(z) = 0, \tag{A.9}
\]
\[
Z_2(0) = 0, \tag{A.10}
\]
\[
Z_2(M_2) = 1. \tag{A.11}
\]
The solution of (A.9)–(A.11) is
\[
Z_2(z) = \exp \left[ \frac{\alpha}{M_2} (1 + i) \right] z + \exp \left[ -\frac{\alpha}{M_2} (1 + i) z \right], \tag{A.12}
\]
Using (A.12), one can obtain
\[
Z_2'(M_2) = \frac{1}{M_2} (1 + i) \theta_2 \tanh[(1 + i) \theta_2] = \frac{1}{M_2} (1 + i) \theta_2 ^2 \frac{\sinh[(1 + i) \theta_2]}{\cosh[(1 + i) \theta_2]} = \frac{1}{M_2} \left( R_{\text{th}}(\theta_2) + i I_{\text{th}}(\theta_2) \right), \tag{A.13}
\]
where \( \theta_2 \) is given by (13c).

Now substituting (A.1), (A.8) and (A.13) back into Eqs. (7)–(9), and extending the three resultant real equations into complex ones with respect to the unknown function \( X(x) \), yield
\[
TX''(x) = \left\{ \frac{K_1}{M_1} R_{\text{th}}(\theta_1) + \frac{K_2}{M_2} R_{\text{th}}(\theta_2) + i \frac{K_1}{M_1} I_{\text{th}}(\theta_1) + \frac{K_2}{M_2} I_{\text{th}}(\theta_2) + \omega S \right\} X, \tag{A.14}
\]
\[
X'(0) = \alpha a X(0) = \alpha a, \tag{A.15}
\]
\[
X(+\infty) = 0. \tag{A.16}
\]
The solution of (A.14)–(A.16) is
\[
X(x) = C_1 e^{-\sigma(x-p_1 i q_1 x)}, \tag{A.17}
\]
where \( a, p_1, q_1 \) are given by Eqs. (10a), (12a) and (12b), respectively; and
\[
C_1 = \frac{\sigma}{\sigma + p_1 + q_1 i}. \tag{A.18}
\]
Substituting (A.18) into (A.1) yields
\[
h(x,t) = A e^{-\alpha q_1 x} \text{Re}[C_1 e^{i(\alpha t - a q_1 x)}] = A e^{-\alpha q_1 x} \text{Re}[C_1 e^{i(\alpha t - a q_1 x + \arg C_1)}] = A[C_1 e^{-\alpha q_1 x} \cos(\alpha t - a q_1 x + \arg C_1)] = AC_1 e^{-\alpha q_1 x} \cos(\alpha t - a q_1 x - \phi_1), \tag{A.19}
\]
where \( C_1 = |C_1| \) and \( \phi_1 = -\arg C_1 \) are defined as Eqs. (15) and (16), respectively.

\[\text{References}\]